# Matrix Algebra

# Addition of Matrices

If $A$ and $B$ are both $m \times n$ matrices then the **sum** of $A$ and $B$, denoted $A + B$, is a matrix obtained by adding ***corresponding elements*** of $A$ and $B$.

add these

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 0 & 4 \\ 2 & 1 & -4 \end{bmatrix}$$

$$A + B = \begin{bmatrix} -2 & -2 & 6 \\ 2 & 0 & -1 \end{bmatrix}$$

$$A + B = B + A$$

**Matrix addition is commutative**

**Matrix addition is associative**

$$A + (B + C) = (A + B) + C$$

# Scalar Multiplication of Matrices

If $A$ is an $m \times n$ matrix and $s$ is a scalar, then we let $kA$ denote the matrix obtained by multiplying every element of $A$ by $k$. This procedure is called **scalar multiplication.**

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 3 \end{bmatrix} \quad 3A = \begin{bmatrix} 3(1) & 3(-2) & 3(2) \\ 3(0) & 3(-1) & 3(3) \end{bmatrix} = \begin{bmatrix} 3 & -6 & 6 \\ 0 & -3 & 9 \end{bmatrix}$$

**PROPERTIES OF SCALAR MULTIPLICATION**

$$k(hA) = (kh)A$$

$$(k+h)A = kA + hA$$

$$k(A+B) = kA + kB$$

The $m \times n$ **zero matrix**, denoted *0*, is the $m \times$ n matrix whose elements are all zeros.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$2 \times 2$

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$1 \times 3$

$$\boldsymbol{A} + \mathbf{0} = \boldsymbol{A}$$

$$\boldsymbol{A} + (-\boldsymbol{A}) = \mathbf{0}$$

$$0(\boldsymbol{A}) = \mathbf{0}$$

# Multiplication of Matrices

**The multiplication of matrices is easier shown than put into words. You multiply the rows of the first matrix with the columns of the second adding products**

**Find *AB***

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$

$$(3)(2) + (-2)(-1) + (1)(-3) = 5$$

**First we multiply across the first row and down the first column adding products. We put the answer in the first row, first column of the answer.**

# Find *AB*

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 7 \\ -1 & 11 \end{bmatrix} \qquad (0)(4) + (4)(3) + (-1)(1) = 11$$

Notice the sizes of *A* and *B* and the size of the product *AB*.

**Now we multiply across the second row and down the second column and we'll put the answer in the second row, second column.**

## **To multiply matrices *A* and *B* look at their dimensions**

$$m \times n \qquad\qquad n \times p$$


If the number of columns of *A* does not equal the number of rows of *B* then the product *AB* is undefined.

# Now let's look at the product BA.

$$B=\begin{bmatrix}2 & 4\\ -1 & 3\\ -3 & 1\end{bmatrix}\qquad A=\begin{bmatrix}3 & -2 & 1\\ 0 & 4 & -1\end{bmatrix}$$

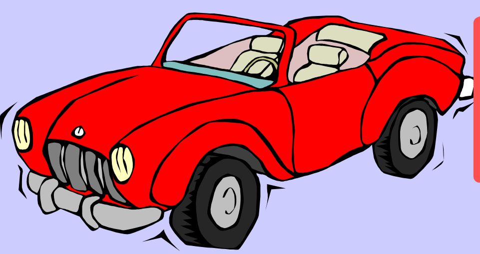
$$(-3)(1)+(1)(-1)=-4$$

$3\times2$ can multiply $2\times3$

size of answer

across third row as we go down third column:

$$BA=\begin{bmatrix}6 & 12 & -2\\ -3 & 14 & -4\\ -9 & 10 & -4\end{bmatrix}$$



**Commuter's Beware!**

$$AB\neq BA$$

**Completely different than *AB*!**

# PROPERTIES OF MATRIX MULTIPLICATION

$$A(BC) = (AB)C$$

$$A(B+C) = AB + AC$$

$$(A+B)C = AC + BC$$

$$AB \neq BA$$

Is it possible for AB = BA ,yes it is possible.

What is $AI$?

$$\begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & 5 \\ 2 & -2 & 3 \end{bmatrix} = A$$

**Multiplying a matrix by the identity gives the matrix back again.**

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What is $IA$?

$$\begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & 5 \\ 2 & -2 & 3 \end{bmatrix} = A$$

$$A = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 1 & 5 \\ 2 & -2 & 3 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# identity matrix

**an $n \times n$ matrix with ones on the main diagonal and zeros elsewhere**

## Can we find a matrix to multiply the first matrix by to get the identity?

$$\begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $A$ be an $n \times n$ matrix. If there exists a matrix $B$ such that $AB = BA = I$ then we call this matrix the **inverse** of $A$ and denote it $A^{-1}$.

If $A$ has an inverse we say that $A$ is **nonsingular**.
If $A^{-1}$ does not exist we say $A$ is **singular**.

**To find the inverse of a matrix we put the matrix A, a line and then the identity matrix. We then perform row operations on matrix A to turn it into the identity. We carry the row operations across and the right hand side will turn into the inverse.**

$$A = \begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -2 & -7 & 0 & 1 \end{array}\right]$$

$$2r_1+r_2 \quad \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right]$$

$$-r_2 \quad \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

$$r_1 - r_2 \quad \left[\begin{array}{cc|cc} 1 & 0 & 7 & 3 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

$$A = \begin{bmatrix} 1 & 3 \\ -2 & -7 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 7 & 3 \\ -2 & -1 \end{bmatrix}$$

**Check this answer by multiplying. We should get the identity matrix if we've found the inverse.**

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# We can use $A^{-1}$ to solve a system of equations

$$x + 3y = 1$$

$$2x + 5y = 3$$

To see how, we can re-write a system of equations as matrices.

$$A\mathbf{x} = \mathbf{b}$$

**coefficient matrix** (A) | **variable matrix** (x) | **constant matrix** (b)

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$

left multiply both sides by the inverse of *A*

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$$

This is just the identity

$$I\mathbf{x} = A^{-1}\mathbf{b}$$

but the identity times a matrix just gives us back the matrix so we have:

$$\mathbf{x} = A^{-1}\mathbf{b}$$

This then gives us a formula for finding the variable matrix: Multiply *A* inverse by the constants.

$$x + 3y = 1$$
$$2x + 5y = 3$$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

find the inverse

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right]$$

$-2r_1+r_2$ $$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

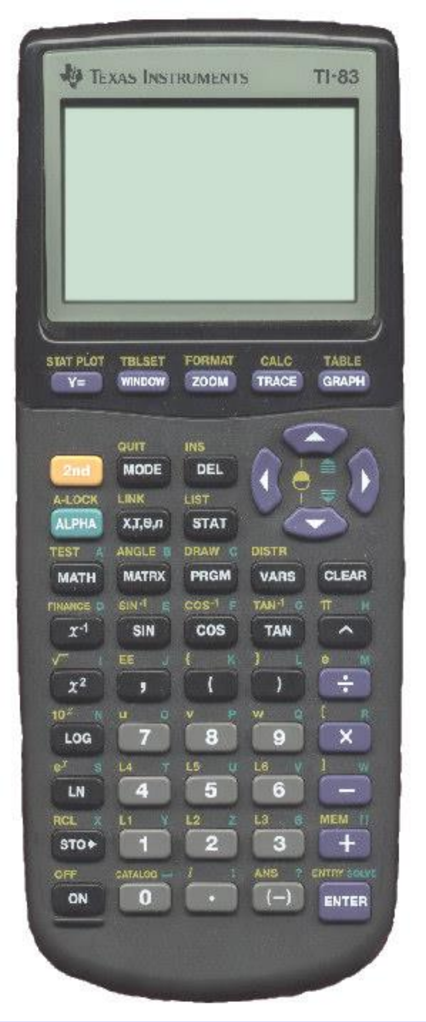
$-r_2$ $$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$r_1-3r_2$ $$\left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$A^{-1}\mathbf{b} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \begin{matrix} \leftarrow x \\ \leftarrow y \end{matrix}$$

This is the answer to the system

**Your calculator can compute inverses and determinants of matrices. To find out how, refer to the manual or click here to check out the website.**



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